Solution Set: Linear Regression

1. Form $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 4 \end{bmatrix}$.

   The coefficients β_0, β_1 for the best fit line $f(x) = \beta_0 + \beta_1 x$ are given by $\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = (X^T X)^{-1} X^T \boldsymbol{y}$.

$$X^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \qquad \Longrightarrow \qquad X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$$

$$\Longrightarrow \qquad (X^T X)^{-1} = \begin{bmatrix} \frac{7}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

$$\Longrightarrow \qquad (X^T X)^{-1} X^T \boldsymbol{y} = \begin{bmatrix} \frac{7}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{14}{10} \\ \frac{9}{10} \end{bmatrix}$$

$$\Longrightarrow \qquad \beta_0 = \frac{14}{10} \text{ and } \beta_1 = \frac{9}{10}.$$

   Thus, the best fit line is given by

$$f(x) = \frac{14}{10} + \frac{9}{10} x$$

   The predicted value for $x = 4$ is $f(4) = \frac{14}{10} + \frac{9}{10} \cdot 4 = 5$.

2. Form $X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

   The coefficients $\beta_0, \beta_1, \beta_2$ for the best fit line $f(x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$ are given by $\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = (X^T X)^{-1} X^T \boldsymbol{y}$.

$$X^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$\Longrightarrow \qquad (X^T X)^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \qquad (X^T X)^{-1} X^T \boldsymbol{y} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\Longrightarrow \qquad \beta_0 = \frac{1}{4} \ , \ \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{2}$$

Thus, the best fit plane is given by

$$f(x_1, x_2) = \frac{1}{4} + \frac{1}{2} x_1 + \frac{1}{2} x_2$$

The predicted value for $x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $f(2, 2) = 2\frac{1}{4}$.