Linear Algebra for Beginners

Open Doors to Great Careers

Richard Han

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PREFACE

Welcome to Linear Algebra for Beginners: Open Doors to Great Careers. This is a first textbook in linear algebra. Be sure to get the companion online course Linear Algebra for Beginners here: https://www.onlinemathtraining.com/linear-algebra/. The online course can be very helpful in conjunction with this book.

The prerequisite for this book and the online course is a basic understanding of algebra.

I want you to succeed and prosper in your career, life, and future endeavors. I am here for you. Visit me at: https://www.onlinemathtraining.com/

1 - INTRODUCTION

Welcome to Linear Algebra for Beginners: Open Doors to Great Careers! My name is Richard Han. This is a first textbook in linear algebra.

Ideal student:

If you're a working professional needing a refresher on linear algebra or a complete beginner who needs to learn linear algebra for the first time, this book is for you. If your busy schedule doesn't allow you to go back to a traditional school, this book allows you to study on your own schedule and further your career goals without being left behind.

If you plan on taking linear algebra in college, this is a great way to get ahead.

If you're currently struggling with linear algebra or have struggled with it in the past, now is the time to master it.

Benefits of studying this book:

After reading this book, you will have refreshed your knowledge of linear algebra for your career so that you can earn a higher salary.

You will have a required prerequisite for lucrative career fields such as Data Science and Artificial Intelligence.

You will be in a better position to pursue a masters or PhD degree in machine learning and data science.

Why Linear Algebra is important:

- Famous uses of linear algebra include:
  - Computer graphics. Matrices are used to rotate figures in three-dimensional space.
  - Cryptography. Messages can be encrypted and decrypted using matrix operations.
  - Machine learning. Eigenvectors can be used to reduce the dimensionality of a data set, using a technique called Principal Component Analysis (PCA).

- Electrical networks. Electrical networks can be solved using systems of linear equations.
- Leontief Input-output model in economics. The necessary outputs of a list of industries can be found using matrix operations.
- Finance. Regression analysis can be used to estimate relationships between financial variables. For example, the relationship between the monthly return to a given stock and the monthly return to the S&P 500 can be estimated using a linear regression model. The model can, in turn, be used to forecast the future monthly return of the given stock.

What my book offers:

In this book, I cover core topics such as:

- **Gaussian Elimination**
- **Vectors**
- **Matrix Algebra**
- **Determinants**
- **Vector Spaces**
- **Subspaces**
- **Span and Linear Independence**
- **Basis and Dimension**

I explain each definition and go through each example step by step so that you understand each topic clearly. Throughout the book, there are practice problems for you to try. Detailed solutions are provided after each problem set.

I hope you benefit from the book.

Best regards,
Richard Han

2 – SOLVING SYSTEMS OF LINEAR EQUATIONS

GAUSSIAN ELIMINATION

In this section, we're going to look at solving systems of linear equations. We're going to look at the process of Gaussian elimination, and it has three things that you can do. The first thing you can do is switch two equations. The second thing is that you can multiply one equation by a nonzero number. The third thing that you can do is add a multiple of one equation to a second equation. This set of three things that you can do is called ***Gaussian elimination***. This will make a lot more sense if we look at some examples. Let's say we had a system of equations like this:

$$x - 2y = 1$$
$$4x + y = 0$$

Here, you have two equations and two variables x and y. So this is a system of two equations in two variables. What we want to do here is try to get rid of the x variable. So let's do -4 times the first equation and add that to the second equation: $-4E_1 + E_2$. If we multiply the first equation by -4 and leave the second equation as it is, we get this:

$$-4x + 8y = -4$$
$$4x + y = 0$$

Adding these two equations, we get:

$$-4x + 8y = -4$$
$$4x + y = 0$$

$$0x + 9y = -4$$

$0x$ is just 0. So we get $9y = -4$. Let's solve for y. Divide both sides by 9. And you get $y = -\frac{4}{9}$.

I want to find what x is; so I'll plug the y-value back in to the first equation:

$$x - 2\left(-\frac{4}{9}\right) = 1$$

Now, simplifying this, we get:

$$x + \frac{8}{9} = 1$$

Solve for x by subtracting $\frac{8}{9}$ from both sides, and I get $x = \frac{1}{9}$.

So $x = \frac{1}{9}$ and $y = -\frac{4}{9}$. That's a solution to our original system of equations.

Let's do another example. Let's say we had this system of equations:

$$x - y = 1$$

$$2x - 2y = 3$$

Let's multiply the first equation by -2 and add to the second equation: $-2E_1 + E_2$.

We get:

$$-2x + 2y = -2$$

$$2x - 2y = 3$$

Adding these two equations, we get:

$$-2x + 2y = -2$$

$$2x - 2y = 3$$

$$0x + 0y = 1$$

The left-hand side equals to 0. So we get $0 = 1$, which is a contradiction. Since we get a contradiction, the original system of equations has no solution.

Let's do one more example. Suppose we have:

$$7x + 5y = 2$$

$$14x + 10y = 4$$

Let's try to get the coefficient of x in the first equation to be -14 so that the $x's$ cancel out when we add both equations. Let's do $-2E_1 + E_2$. We get:

$$-14x - 10y = -4$$

$$14x + 10y = 4$$

Adding the two equations, we get:

$$0x + 0y = 0$$

And so: $0 = 0$. That doesn't really tell me anything. If you look back at the original system of equations, notice the second equation is just twice the first equation. So, really, we only have just one equation, the first equation. The second equation is redundant. So, all we have is $7x + 5y = 2$. Note that y can be anything. Let y be some parameter t. Let's plug that in for y and solve for x:

$$7x + 5t = 2$$

Bring the $5t$ to the other side: $7x = 2 - 5t$

Dividing both sides by 7: $x = \frac{2}{7} - \frac{5}{7}t$

So, the set of all solutions is going to be the set of all pairs $\left(\frac{2}{7} - \frac{5}{7}t, t\right)$ where t is any real number.

GAUSSIAN ELIMINATION AND ROW ECHELON FORM

For a system of 3 equations and 3 variables, we want to solve in a similar fashion by getting rid of the variables one by one until we have a triangular shape. Let's look at an example.

Suppose we have the following system of equations:

$$x + y + z = 0$$
$$-x + 2y + 3z = 1$$
$$3x - 3y + z = -1$$

We have three equations and three variables x, y, and z. Notice that, in the second equation, we have a $-x$; and, if we were to add that to the first equation, the x terms would cancel out. So let's take the first equation E_1 and add that to the second equation E_2. Replace the second equation with $E_1 + E_2$ like this:

$$x + y + z = 0$$
$$3y + 4z = 1$$
$$3x - 3y + z = -1$$

Now, let's try to get rid the x term in the third equation by multiplying the first equation by -3 and adding the result to the third equation like this: $-3E_1 + E_3$. Replace the third equation with $-3E_1 + E_3$ to get:

$$x + y + z = 0$$
$$3y + 4z = 1$$
$$-6y - 2z = -1$$

Let's try to get rid of the y term in the third equation by doing $2E_2 + E_3$ and replacing the third equation by the result:

$$x + y + z = 0$$
$$3y + 4z = 1$$
$$6z = 1$$

Look at the $6z$ in the third equation. We want the coefficient of z to be 1. So divide the third equation by 6: $\frac{1}{6}E_3$. Replace the third equation by the result to get:

$$x + y + z = 0$$
$$3y + 4z = 1$$
$$z = \frac{1}{6}$$

Look at the coefficient of y in the second equation. We want that to be 1. So let's divide the second equation by 3: $\frac{1}{3}E_2$. Replace the second equation by the result to get:

$$x + y + z = 0$$
$$y + \frac{4}{3}z = \frac{1}{3}$$
$$z = \frac{1}{6}$$

Now, notice that all the coefficients of the leading variables in each equation are 1. When you have a triangular shape like the above and all the leading coefficients are 1, then we say that the system of equations is in ***row echelon form***.

Let's do another example. Suppose we have the following system of equations:

$$x - 2y + 5z = 2$$
$$3x + 2y - z = -2$$

Let's try to get rid of the x term in the second equation by performing: $-3E_1 + E_2$. Replace the second equation by the result to get:

$$x - 2y + 5z = 2$$
$$8y - 16z = -8$$

Looking at the second equation, notice that z can be anything. So let $z = t$, where t is a free variable. Plug in t for z in the second equation and solve for y:

$$8y - 16t = -8$$
$$8y = 16t - 8$$
$$y = 2t - 1$$

So we have y in terms of t. We have z and y in terms of t. Now, we want to solve for x. Let's use the first equation $x - 2y + 5z = 2$. Plug in what we got for y and z and solve for x:

$$x - 2(2t - 1) + 5t = 2$$
$$x - 4t + 2 + 5t = 2$$
$$x + 2 + t = 2$$
$$x + t = 0$$

$$x = -t$$

So, we have $x = -t, y = 2t - 1, z = t$. Any point $(-t, 2t - 1, t)$, where $t \in \mathbb{R}$, is a solution to the original system of equations.

Let's do one more example. Suppose we have the following system of equations:

$$x + y - z = 0$$
$$x - y + z = 1$$
$$2x + y - z = 0$$

If we take the first equation and subtract the second equation, we can get rid of the x-term. So let's do $E_1 - E_2$. Replace the second equation with the result:

$$x + y - z = 0$$
$$2y - 2z = -1$$
$$2x + y - z = 0$$

Now, look at the first equation and the third equation; we want to get rid of the x-term. So, let's do $-2E_1 + E_3$.

$$x + y - z = 0$$
$$2y - 2z = -1$$
$$-y + z = 0$$

Let's look at the second and third equations; let's get rid of the y-term. Perform $E_2 + 2E_3$.

$$x + y - z = 0$$
$$2y - 2z = -1$$
$$0 = -1$$

We arrive at a contradiction. Therefore, the original set of equations has no solution.

PROBLEM SET: GAUSSIAN ELIMINATION

Solve the system of linear equations using Gaussian elimination.

1. $$\begin{aligned} x+2y&=0 \\ -x+y&=10 \end{aligned}$$

2. $$\begin{aligned} 3x-y&=3 \\ -4x+11y&=7 \end{aligned}$$

3. $$\begin{aligned} 8x-5y&=20 \\ -16x+10y&=-40 \end{aligned}$$

4. $$\begin{aligned} 2x+y&=13 \\ -4x-2y&=4 \end{aligned}$$

5. $$\begin{aligned} x-y-z&=1 \\ 2x+y+3z&=0 \\ 3x-y+z&=-1 \end{aligned}$$

6. $$\begin{aligned} x+2y+2z&=4 \\ -y+z&=-1 \\ x+y&=8 \end{aligned}$$

7. $$\begin{aligned} x-y-z&=3 \\ x-10y+10z&=0 \end{aligned}$$

SOLUTION SET: GAUSSIAN ELIMINATION

1. $\begin{matrix} x+2y=0 \\ -x+y=10 \end{matrix}$

Add Equation 1 to Equation 2 to get our new Equation 2.

$$x + 2y = 0$$
$$3y = 10$$

Solve the second equation for y to get $y = \frac{10}{3}$. Then plug this y value back into equation 1 to get $x + 2\left(\frac{10}{3}\right) = 0$. Solving for x gives $x = -\frac{20}{3}$.

2. $\begin{matrix} 3x-y=3 \\ -4x+11y=7 \end{matrix}$

Take 4 times Equation 1 and add to 3 times Equation 2 to get our new Equation 2.

$$3x - y = 3$$
$$29y = 33$$

Solve for y in the second equation to get y=$\frac{33}{29}$. Plug this value into Equation 1 to solve for x.

$3x - \frac{33}{29}$=3.

$$3x = 3 + \frac{33}{29}$$
$$3x = \frac{120}{29}$$
$$x = \frac{40}{29}$$

3. $\begin{matrix} 8x-5y=20 \\ -16x+10y=-40 \end{matrix}$

Take 2 times Equation 1 and add to Equation 2 to get the new second equation.

$$8x - 5y = 20$$
$$0 = 0$$

Now, y is a free variable. So let y=t, where t is a parameter. The first equation gives us $8x - 5t = 20$. Solve for x to get $x = \frac{5}{8}t + \frac{5}{2}$.

4. $\begin{array}{c} 2x+y=13 \\ -4x-2y=4 \end{array}$

Add 2 times Equation 1 to Equation 2 to get our new Equation 2.

$$2x + y = 13$$
$$0 = 30$$

Since the second equation is a contradiction, there is no solution.

5. $\begin{array}{c} x-y-z=1 \\ 2x+y+3z=0 \\ 3x-y+z=-1 \end{array}$

Take -2 times the first equation and add to the second equation to get our new second equation.

$$x - y - z = 1$$
$$3y + 5z = -2$$
$$3x - y + z = -1$$

Now, take -3 times the first equation and add to the third equation to get our new third equation.

$$x - y - z = 1$$
$$3y + 5z = -2$$
$$2y + 4z = -4$$

Now, take -2Eq.2+3Eq.3 to get our new Eq.3.

$$x - y - z = 1$$
$$3y + 5z = -2$$
$$2z = -8$$

Solve the third equation for z. z=-4. Plug this into the second equation and solve for y.

$$3y - 20 = -2$$

$$y = 6$$

Plug the z and y values into the first equation and solve for x.

$$x - 6 - (-4) = 1$$

$$x = 3$$

6. $\begin{array}{c} x+2y+2z=4 \\ -y+z=-1 \\ x+y=8 \end{array}$

Add the first equation to 2 times the second equation to get our new second equation.

$$x + 2y + 2z = 4$$
$$x + 4z = 2$$
$$x + y = 8$$

Take -2Eq.3+Eq.1 to get our new Eq.3.

$$x + 2y + 2z = 4$$
$$x + 4z = 2$$
$$-x + 2z = -12$$

Take Eq.2+Eq.3 to get our new Eq.3.

$$x + 2y + 2z = 4$$
$$x + 4z = 2$$
$$6z = -10$$

Now, solve for z in the third equation. $z = -\frac{5}{3}$.

Plug the z value into the second equation and solve for x.

$$x - \frac{20}{3} = 2$$

$$x = \frac{26}{3}$$

Plug the x and z values into the first equation and solve for y.

$$\frac{26}{3} + 2y - \frac{10}{3} = 4$$

$$2y + \frac{16}{3} = 4$$

$$y = -\frac{2}{3}$$

7. $\begin{matrix} x-y-z=3 \\ x-10y+10z=0 \end{matrix}$

Let's do Eq.1-Eq.2 to get our new Eq. 2.

$$x - y - z = 3$$
$$9y - 11z = 3$$

Now, z is a free variable, so let z=t. Then plug that value into z in the second equation.

$$9y - 11t = 3$$

Solve for y.

$$y = \frac{11}{9}t + \frac{1}{3}$$

Plug the values we got for y and z into the first equation and solve for x.

$$x - (\frac{11}{9}t + \frac{1}{3}) - t = 3$$

$$x - \frac{20}{9}t - \frac{1}{3} = 3$$

$$x = \frac{20}{9}t + \frac{10}{3}$$

ELEMENTARY ROW OPERATIONS

We can rewrite a system of equations using a matrix. For example, look at this system of equations:

$$x + y + z = 0$$

$$-x + 2y + 3z = 1$$

$$3x - 3y + z = -1$$

We can write a matrix that encapsulates this system of equations. We look at the coefficients of the variables in this system of equations. For the first equation, the coefficients of the variables are 1, 1, and 1. On the right hand side, we have the constant 0. So, for the first row of the matrix, we have:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ & & & \end{bmatrix}$$

Now, move on to the second equation. The coefficients are -1, 2, and 3. The constant on the right hand side is 1. So fill in the second row for the matrix like so:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 3 & 1 \\ & & & \end{bmatrix}$$

Let's move on to the third equation. The coefficients are $3, -3$, and 1. The constant is -1. So fill in the third row of the matrix like so:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 3 & 1 \\ 3 & -3 & 1 & -1 \end{bmatrix}$$

This matrix that we just formed is called the ***augmented matrix***.

Now, we can solve the system of equations using the same three operations we used earlier. Instead of performing operations on equations, we can perform operations on rows. The operations are called ***elementary row operations***. The first thing you can do is switch two rows. The second thing you can do is multiply one row by a nonzero number. The third thing you can do is add a multiple of one row to a second row. Here is the list of the three elementary row operations:

1. Switch two rows.
2. Multiply one row by a nonzero number.
3. Add a multiple of one row to a second row.

These are exactly the same three steps that you saw earlier in Gaussian elimination.

Let's look at the augmented matrix we had earlier.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 3 & 1 \\ 3 & -3 & 1 & -1 \end{bmatrix}$$

Let's look at the first two rows. If we were to add those two rows, the 1 and the -1 would cancel out. So let's add rows 1 and 2 to get a new row 2: $R1 + R2 \longrightarrow R2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 3 & -3 & 1 & -1 \end{bmatrix}$$

Look at the third row; we want to get rid of the 3. So do $-3R1 + R3 \longrightarrow R3$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & -6 & -2 & -1 \end{bmatrix}$$

Let's focus on the second and third rows; we want to get rid of the -6 in the third row. So let's do $2R2 + R3 \longrightarrow R3$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 6 & 1 \end{bmatrix}$$

Look at the third row; we want the leading coefficient 6 to be 1. So divide the third row by 6 to get a new third row: $\frac{1}{6}R3 \longrightarrow R3$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}$$

Look at the second row; we want the coefficient 3 to be 1. So perform $\frac{1}{3}R2 \longrightarrow R2$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/3 & 1/3 \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}$$

Notice the triangular shape of the matrix; all the leading coefficients in each row are 1. Furthermore, the leading coefficient of any row is to the right of the leading coefficient in the previous row. Also, any row of all zeroes is at the bottom of the matrix (in our example, there is no row of all zeroes). Since these three conditions of row-echelon form are satisfied, our matrix is in row-echelon form.

ELEMENTARY ROW OPERATIONS: ADDITIONAL EXAMPLE

Let's do an additional example. Suppose we had the following system of equations:

$$y + z = 0$$

$$x - y - z = 1$$

$$2x + 2y - z = 3$$

Let's write the corresponding augmented matrix. Note that, in the first equation, there is no x term; so the coefficient for x is 1. So the coefficients for the first equation are $0, 1,$ and 1. The augmented matrix so far looks like this:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ & & & \end{bmatrix}$$

Filling in the second and third rows, we get:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & 2 & -1 & 3 \end{bmatrix}$$

The first coefficient in the first row is 0, and we want a 1 there instead. So let's switch the first row with the second row: $R1 \leftrightarrow R2$

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 2 & -1 & 3 \end{bmatrix}$$

Let's look at the third row. We want to get rid of the first 2 and make it a 0. So let's do $-2R1 + R3 \longrightarrow R3$.

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 1 & 1 \end{bmatrix}$$

Let's get rid of the 4 in the third row by doing $-4R2 + R3 \longrightarrow R3$.

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

Let's make the -3 in the third row a 1 by doing $-\frac{1}{3}R3 \longrightarrow R3$.

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

Notice the triangular shape and that all the leading coefficients are 1. So this matrix is in row echelon form.

PROBLEM SET: ELEMENTARY ROW OPERATIONS

Solve the system of linear equations using an augmented matrix and elementary row operations.

1. $$\begin{gathered} 7x+7y-z=3 \\ x+y+z=-1 \\ -x-y+3z=0 \end{gathered}$$

2. $$\begin{gathered} 3x-y+11z=7 \\ 7x+y+7z=-3 \\ 14x+2y+14z=-6 \end{gathered}$$

3. $$\begin{gathered} x-11y-z=8 \\ 8x+y-z=2 \\ -7x-12y=-12 \end{gathered}$$

4. $$\begin{gathered} x-z=2 \\ x+y+z=-3 \\ x-y=0 \end{gathered}$$

5. $$\begin{gathered} x+z=8 \\ y+z=-10 \end{gathered}$$

6. $$\begin{gathered} x-y+z=9 \\ 9y-9z=3 \end{gathered}$$

SOLUTION SET: ELEMENTARY ROW OPERATIONS

1. $7x+7y-z=3$
$x+y+z=-1$
$-x-y+3z=0$

$$\begin{bmatrix} 7 & 7 & -1 & 3 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 3 & 0 \end{bmatrix}$$ $-7R_2+R_1 \to R_2$

$$\begin{bmatrix} 7 & 7 & -1 & 3 \\ 0 & 0 & -8 & 10 \\ -1 & -1 & 3 & 0 \end{bmatrix}$$ $-7R_3+R_1 \to R_3$

$$\begin{bmatrix} 7 & 7 & -1 & 3 \\ 0 & 0 & -8 & 10 \\ 0 & 0 & 20 & 3 \end{bmatrix}$$ $-\frac{1}{8}R_2 \to R_2$

$$\begin{bmatrix} 7 & 7 & -1 & 3 \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 20 & 3 \end{bmatrix}$$ $-20R_2+R_3 \to R_3$

$$\begin{bmatrix} 7 & 7 & -1 & 3 \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 28 \end{bmatrix}$$

The last row gives that 0=28, which is a contradiction. So there is no solution.

2. $3x-y+11z=7$
$7x+y+7z=-3$
$14x+2y+14z=-6$

$$\begin{bmatrix} 3 & -1 & 11 & 7 \\ 7 & 1 & 7 & -3 \\ 14 & 2 & 14 & -6 \end{bmatrix}$$ $7R_1-3R_2 \to R_2$

$$\begin{bmatrix} 3 & -1 & 11 & 7 \\ 0 & -10 & 56 & 58 \\ 14 & 2 & 14 & -6 \end{bmatrix}$$ $\frac{1}{2}R_3 \to R_3$

$$\begin{bmatrix} 3 & -1 & 11 & 7 \\ 0 & -10 & 56 & 58 \\ 7 & 1 & 7 & -3 \end{bmatrix}$$ $7R_1-3R_3 \to R_3$

$$\begin{bmatrix} 3 & -1 & 11 & 7 \\ 0 & -10 & 56 & 58 \\ 0 & -10 & 56 & 58 \end{bmatrix}$$ $R_2-R_3 \to R_3$

$$\begin{bmatrix} 3 & -1 & 11 & 7 \\ 0 & -10 & 56 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the second row, we have $-10y + 56z = 58$. Since z is a free variable, let z=t. Then $-10y + 56t = 58$. Solve for y.

$$y = \frac{28}{5}t - \frac{29}{5}$$

Now, the first row tells us

$3x - y + 11z = 7$. Plug in the values we got for y and z. Then solve for x.

$$3x - \left(\frac{28}{5}t - \frac{29}{5}\right) + 11t = 7$$

$$x = -\frac{9}{5}t + \frac{2}{5}$$

So $x = -\frac{9}{5}t + \frac{2}{5}, y = \frac{28}{5}t - \frac{29}{5}, z = t, t \in \mathbb{R}$.

3. $x-11y-z=8$
$8x+y-z=2$
$-7x-12y=-12$

$$\begin{bmatrix} 1 & -11 & -1 & 8 \\ 8 & 1 & -1 & 2 \\ -7 & -12 & 0 & -12 \end{bmatrix} \quad -8R_1+R_2 \to R_2$$

$$\begin{bmatrix} 1 & -11 & -1 & 8 \\ 0 & 89 & 7 & -62 \\ -7 & -12 & 0 & -12 \end{bmatrix} \quad 7R_1+R_3 \to R_3$$

$$\begin{bmatrix} 1 & -11 & -1 & 8 \\ 0 & 89 & 7 & -62 \\ 0 & -89 & -7 & 44 \end{bmatrix} \quad R_2+R_3 \to R_3$$

$$\begin{bmatrix} 1 & -11 & -1 & 8 \\ 0 & 89 & 7 & -62 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

The last row tells us that 0=-18, which is a contradiction. Therefore, there is no solution.

4. $x-z=2$
$x+y+z=-3$
$x-y=0$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & -3 \\ 1 & -1 & 0 & 0 \end{bmatrix} \quad R_1-R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & -2 & 5 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad R_1\text{-}R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & -2 & 5 \\ 0 & 1 & -1 & 2 \end{bmatrix} \qquad R_2\text{+}R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & -2 & 5 \\ 0 & 0 & -3 & 7 \end{bmatrix}$$

The third row tells us -3z=7. So z=$-\frac{7}{3}$.

The second row tells us –y-2z=5.

$-y - 2\left(-\frac{7}{3}\right) = 5.$

$y = -\frac{1}{3}.$

The first row tells us x-z=2. So $x - \left(-\frac{7}{3}\right) = 2.$

$x = -\frac{1}{3}$

5. $\begin{matrix} x+z=8 \\ y+z=-10 \end{matrix}$

$$\begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -10 \end{bmatrix}$$

The second row tells us y+z=-10. Z is a free variable, so let z=t. Then y+t=-10. Y=-t-10.

The first row tells us x+z=8. So x+t=8. X=-t+8.

$$x = -t + 8, y = -t - 10, z = t, t \in \mathbb{R}$$

6. $\begin{matrix} x-y+z=9 \\ 9y-9z=3 \end{matrix}$

$$\begin{bmatrix} 1 & -1 & 1 & 9 \\ 0 & 9 & -9 & 3 \end{bmatrix}$$

The second row tells us 9y-9z=3. Let z=t. Then $y = t + \frac{1}{3}$.

The first row tells us $x - y + z = 9$.

$$x - \left(t + \frac{1}{3}\right) + t = 9$$

$$x = \frac{28}{3}$$

$$x = \frac{28}{3}, y = t + \frac{1}{3}, z = t, t \in \mathbb{R}$$

SUMMARY: SOLVING SYSTEMS OF LINEAR EQUATIONS

- Gaussian elimination consists of the following three actions:
  1. Switch two equations.
  2. Multiply one equation by a nonzero number.
  3. Add a multiple of one equation to a second equation.

- Elementary row operations consist of the following three actions:
  1. Switch two rows.
  2. Multiply one row by a nonzero number.
  3. Add a multiple of one row to a second row.

- A matrix is in row echelon form if the following three conditions hold:
  1. Any row of all zeroes is at the bottom of the matrix.
  2. The leading coefficient of any row is to the right of the leading coefficient in the previous row.
  3. All leading coefficients are 1.

3 – VECTORS

VECTOR OPERATIONS AND LINEAR COMBINATIONS

We will now learn about vector operations and linear combinations. What is a vector? A vector is a list of real numbers. Let's look at an example:

$$\boldsymbol{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

This is a vector in $\mathbb{R}^2$. $\mathbb{R}^2$ symbolizes all pairs of real numbers. Consider another example:

$$\boldsymbol{u} = \begin{bmatrix} 0 \\ 3 \\ -22 \end{bmatrix}$$

Notice this vector has three entries; so this is a vector in $\mathbb{R}^3$.

Now, we want to be able to add two vectors. We can add two vectors $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ by adding their corresponding entries. For example, suppose $\boldsymbol{u_1} = \begin{bmatrix} 0 \\ 3 \\ -22 \end{bmatrix}$ and $\boldsymbol{u_2} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$. To add $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$, form the vector that you get when you add the corresponding entries in each vector.

$$\boldsymbol{u_1} + \boldsymbol{u_2} = \begin{bmatrix} 7 \\ 5 \\ -17 \end{bmatrix}$$

So, if we want to add two vectors, we just add their corresponding entries. The two vectors need to have the same number of entries.

We can also multiply a vector by a scalar c. A scalar is just a real number. Let's do an example.

Suppose $c = 8$ and $\boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. We want to find $c\boldsymbol{v}$. To multiply a vector by a scalar, we just multiply each entry of the vector by the scalar.

$$c\boldsymbol{v} = 8\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}8\\-8\\16\end{bmatrix}$$

If we have a set of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}$ and scalars $c_1, c_2, \dots, c_k$, then $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$ is called a ***linear combination*** of $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}$ and $c_1, \dots, c_k$ are called ***weights***. Let's look at an example.

Suppose $\boldsymbol{v_1} = \begin{bmatrix}1\\1\\2\end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix}0\\-1\\0\end{bmatrix}, \boldsymbol{v_3} = \begin{bmatrix}2\\2\\2\end{bmatrix}$ and $c_1 = 1, c_2 = 3, c_3 = -5$. Then $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3}$ is a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ with weights $1, 3, -5$. We can simplify the linear combination:

$$\begin{aligned} c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3} &= 1\begin{bmatrix}1\\1\\2\end{bmatrix} + 3\begin{bmatrix}0\\-1\\0\end{bmatrix} + (-5)\begin{bmatrix}2\\2\\2\end{bmatrix} \\ &= \begin{bmatrix}1\\1\\2\end{bmatrix} + \begin{bmatrix}0\\-3\\0\end{bmatrix} + \begin{bmatrix}-10\\-10\\-10\end{bmatrix} \\ &= \begin{bmatrix}-9\\-12\\-8\end{bmatrix} \end{aligned}$$

We can also multiply a vector $\boldsymbol{x}$ by a matrix A. For example, suppose $A = \begin{bmatrix}1 & 0 & 2\\3 & 1 & -1\\2 & 2 & 0\end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix}5\\2\\4\end{bmatrix}$. We can multiply the matrix A to the vector $\boldsymbol{x}$ as follows:

$$A\boldsymbol{x} = \begin{bmatrix}1 & 0 & 2\\3 & 1 & -1\\2 & 2 & 0\end{bmatrix}\begin{bmatrix}5\\2\\4\end{bmatrix}$$

To multiply this out, look at the first row of A. We multiply the first entry in the first row with the first entry of , multiply the second entry in the first row with the second entry of $\boldsymbol{x}$, multiply the third entry of the first row with the third entry of $\boldsymbol{x}$, and add the results to get the first entry.:

$$\begin{bmatrix}1 & 0 & 2\\3 & 1 & -1\\2 & 2 & 0\end{bmatrix}\begin{bmatrix}5\\2\\4\end{bmatrix} = \begin{bmatrix}1\cdot 5 + 0\cdot 2 + 2\cdot 4\\ \\ \end{bmatrix} = \begin{bmatrix}13\\ \\ \end{bmatrix}$$

For the second entry, we do the same thing but with the second row of A:

$$\begin{bmatrix}1 & 0 & 2\\3 & 1 & -1\\2 & 2 & 0\end{bmatrix}\begin{bmatrix}5\\2\\4\end{bmatrix} = \begin{bmatrix}1\cdot 5 + 0\cdot 2 + 2\cdot 4\\ 3\cdot 5 + 1\cdot 2 + (-1)\cdot 4\\ \end{bmatrix} = \begin{bmatrix}13\\13\\ \end{bmatrix}$$

For the third entry, we do the same thing but with the third row of A:

$$\begin{bmatrix}1 & 0 & 2\\3 & 1 & -1\\2 & 2 & 0\end{bmatrix}\begin{bmatrix}5\\2\\4\end{bmatrix} = \begin{bmatrix}1\cdot 5 + 0\cdot 2 + 2\cdot 4\\ 3\cdot 5 + 1\cdot 2 + (-1)\cdot 4\\ 2\cdot 5 + 2\cdot 2 + 0\cdot 4\end{bmatrix} = \begin{bmatrix}13\\13\\14\end{bmatrix}$$

PROBLEM SET: VECTOR OPERATIONS AND LINEAR COMBINATIONS

1. Add the two vectors.
   - ***a.*** $\boldsymbol{u}=\begin{bmatrix}1\\0\end{bmatrix}$ $\quad \boldsymbol{v}=\begin{bmatrix}-2\\1\end{bmatrix}$
   - ***b.*** $\boldsymbol{u}=\begin{bmatrix}1\\1\end{bmatrix}$ $\quad \boldsymbol{v}=\begin{bmatrix}-1\\3\end{bmatrix}$
   - ***c.*** $\boldsymbol{u}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad \boldsymbol{v}=\begin{bmatrix}2\\-3\\10\end{bmatrix}$

2. Multiply the vector by the given scalar.
   - ***a.*** $\boldsymbol{u}=\begin{bmatrix}-1\\1\end{bmatrix}$ $\quad c=8$
   - ***b.*** $\boldsymbol{u}=\begin{bmatrix}2\\2\end{bmatrix}$ $\quad c=-3$
   - ***c.*** $\boldsymbol{u}=\begin{bmatrix}-9\\9\\18\end{bmatrix}$ $\quad c=-1$

3. Simplify the linear combination of vectors $c_1\mathbf{v_1}+c_2\mathbf{v_2}+c_3\mathbf{v_3}$.
   - ***a.*** $\boldsymbol{v_1}=\begin{bmatrix}0\\1\end{bmatrix}$ $\quad \boldsymbol{v_2}=\begin{bmatrix}-3\\6\end{bmatrix}$ $\quad \boldsymbol{v_3}=\begin{bmatrix}11\\9\end{bmatrix}$

     $c_1=-1$ $\quad c_2=3$ $\quad c_3=4$.
   - ***b.*** $\boldsymbol{v_1}=\begin{bmatrix}2\\2\end{bmatrix}$ $\quad \boldsymbol{v_2}=\begin{bmatrix}-1\\2\end{bmatrix}$ $\quad \boldsymbol{v_3}=\begin{bmatrix}7\\7\end{bmatrix}$

     $c_1=3$ $\quad c_2=-3$ $\quad c_3=0$.
   - ***c.*** $\boldsymbol{v_1}=\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\quad \boldsymbol{v_2}=\begin{bmatrix}1\\1\\2\end{bmatrix}$ $\quad \boldsymbol{v_3}=\begin{bmatrix}9\\8\\4\end{bmatrix}$

     $c_1=1$ $\quad c_2=6$ $\quad c_3=-1$.
   - ***d.*** $\boldsymbol{v_1}=\begin{bmatrix}4\\-4\\4\end{bmatrix}$ $\quad \boldsymbol{v_2}=\begin{bmatrix}1\\1\\0\end{bmatrix}$ $\quad \boldsymbol{v_3}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

     $c_1=0$ $\quad c_2=1$ $\quad c_3=8$.

4. Find A**v**.
   - ***a.*** $\boldsymbol{v}=\begin{bmatrix}9\\-9\end{bmatrix}$ $\quad A=\begin{bmatrix}-2 & 3\\4 & 2\end{bmatrix}$
   - ***b.*** $\boldsymbol{v}=\begin{bmatrix}8\\-4\end{bmatrix}$ $\quad A=\begin{bmatrix}1 & 0\\-1 & 0\end{bmatrix}$

c. $v = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ -7 & 7 & 14 \end{bmatrix}$

d. $v = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

SOLUTION SET: VECTOR OPERATIONS AND LINEAR COMBINATIONS

1. a. $\boldsymbol{u}+\boldsymbol{v}=\begin{bmatrix}-1\\1\end{bmatrix}$

   b. $\boldsymbol{u}+\boldsymbol{v}=\begin{bmatrix}0\\4\end{bmatrix}$

   c. $\boldsymbol{u}+\boldsymbol{v}=\begin{bmatrix}3\\-2\\11\end{bmatrix}$

2. a. $c\boldsymbol{u}=\begin{bmatrix}-8\\8\end{bmatrix}$

   b. $c\boldsymbol{u}=\begin{bmatrix}-6\\-6\end{bmatrix}$

   c. $c\boldsymbol{u}=\begin{bmatrix}9\\-9\\-18\end{bmatrix}$

3. a. $c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}+c_3\boldsymbol{v_3}=\begin{bmatrix}35\\53\end{bmatrix}$

   b. $c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}+c_3\boldsymbol{v_3}=\begin{bmatrix}9\\0\end{bmatrix}$

   c. $c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}+c_3\boldsymbol{v_3}=\begin{bmatrix}-3\\-2\\9\end{bmatrix}$

   d. $c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}+c_3\boldsymbol{v_3}=\begin{bmatrix}1\\1\\0\end{bmatrix}$

4. a. $A\boldsymbol{v}=\begin{bmatrix}-45\\18\end{bmatrix}$

   b. $A\boldsymbol{v}=\begin{bmatrix}8\\-8\end{bmatrix}$

   c. $A\boldsymbol{v}=\begin{bmatrix}2\\0\\21\end{bmatrix}$

   d. $A\boldsymbol{v}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$

VECTOR EQUATIONS AND THE MATRIX EQUATION Ax=b

We're now ready to look at vector equations and the matrix equation of the form $A\boldsymbol{x} = \boldsymbol{b}$. Recall the system of equations

$$x + y + z = 0$$

$$-x + 2y + 3z = 1$$

$$3x - 3y + z = -1$$

Look at the x variables in the system of equations and think of the x terms as one column. Similarly, think of the y terms as one column, think of the z terms as one column, and think of the constant terms on the right hand sides as one column. We can rewrite the system of equations as:

$$\begin{bmatrix} x \\ -x \\ 3x \end{bmatrix} + \begin{bmatrix} y \\ 2y \\ -3y \end{bmatrix} + \begin{bmatrix} z \\ 3z \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Now, factor out the x from the first column, the y from the second column, and the z from the third column like this:

$$x\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

So to ask if there is a solution to the system of equations is the same as asking if we can write $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ as a linear combination of the column vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. In this case, x, y, z are the weights of the linear combination.

Now, let's define the span. The span of a set of vectors is the set of all linear combinations of those vectors. Thus, we want to know if the vector $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ lies in the $span\left\{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right\}$. Note that the vector equation

$$x\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

can be written as a matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

To see this, multiply out the left hand side to get:

$$\begin{bmatrix} x+y+z \\ -x+2y+3z \\ 3x-3y+z \end{bmatrix}$$

We can rewrite this as the sum of the column vectors consisting of the variables:

$$\begin{bmatrix} x \\ -x \\ 3x \end{bmatrix} + \begin{bmatrix} y \\ 2y \\ -3y \end{bmatrix} + \begin{bmatrix} z \\ 3z \\ z \end{bmatrix}$$

Factoring out the variables, we get:

$$x\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

So, we get the linear combination of the coefficient vectors we had earlier. So our original system of equations can be rewritten as a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ where A is the coefficient matrix, $\boldsymbol{x}$ is the column vector of weights $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\boldsymbol{b}$ is our column vector of constants $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

LINEAR INDEPENDENCE

We will now introduce the notion of linear independence. Let $\boldsymbol{v_1}, \dots, \boldsymbol{v_k} \in \mathbb{R}^n$. Then the set $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_k}\}$ is ***linearly independent*** just in case the vector equation $c_1\boldsymbol{v_1} + \cdots + c_k\boldsymbol{v_k} = 0$ has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, the set is said to be ***linearly dependent***. Note the vector equation $c_1\boldsymbol{v_1} + \cdots + c_k\boldsymbol{v_k} = 0$ can be rewritten as $A\boldsymbol{x} = \boldsymbol{0}$, where $A = [\boldsymbol{v_1} \dots \boldsymbol{v_k}]$ and $\boldsymbol{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$. In A, the vector $\boldsymbol{v_1}$ forms the first column, $\boldsymbol{v_2}$ forms the second column, etc.

LINEAR INDEPENDENCE: EXAMPLE 1

Let's do an example.

Determine if the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is linearly independent.

$$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}, \boldsymbol{v_3} = \begin{bmatrix} 13 \\ -6 \\ 7 \end{bmatrix}$$

Form the matrix with $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ as columns:

$$\begin{bmatrix} 1 & 3 & 13 \\ -1 & 4 & -6 \\ 0 & 7 & 7 \end{bmatrix}$$

Now, set an arbitrary linear combination of the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ equal to $\boldsymbol{0}$ in matrix form:

$$\begin{bmatrix} 1 & 3 & 13 \\ -1 & 4 & -6 \\ 0 & 7 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We want to know if this system of equations has only the trivial solution (in other words, $c_1 = c_2 = c_3 = 0$). Let's form the augmented matrix:

$$\begin{bmatrix} 1 & 3 & 13 & 0 \\ -1 & 4 & -6 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix}$$

Let's try to do a bunch of row operations on this augmented matrix to solve for the solution. If we get the trivial solution, then we know that the trivial solution would be the only solution, and the original set of vectors would be linearly independent. If we find that this system of equations has nontrivial solutions, then we know that the original set of vectors is linearly dependent.

Looking at the first two rows, let's do $R1 + R2 \longrightarrow R2$:

$$\begin{bmatrix} 1 & 3 & 13 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix}$$

Look at the second and third rows. Let's do $R2 - R3 \longrightarrow R3$:

$$\begin{bmatrix} 1 & 3 & 13 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Looking at the leading coefficient 7 in the second row, let's make that a 1 by doing $\frac{1}{7}R2 \longrightarrow R2$:

$$\begin{bmatrix} 1 & 3 & 13 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Looking at the second row, notice that the value for the variable c_3 is free; it can be anything. Let $c_3 = t$, a free parameter. From the second row, we know that $c_2 + c_3 = 0$. Plugging in $c_3 = t$, solve for c_2:

$$c_2 + t = 0$$

$$c_2 = -t$$

From the first row, we know that $c_1 + 3c_2 + 13c_3 = 0$. Plugging in $c_2 = -t$ and $c_3 = t$, we get

$$c_1 + 3(-t) + 13t = 0$$

Solving for c_1, we get:

$$c_1 = -10t$$

Now we have $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -10t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -10 \\ -1 \\ 1 \end{bmatrix}$. Our set of solutions consists of all vectors of the form $t\begin{bmatrix} -10 \\ -1 \\ 1 \end{bmatrix}$. So there are many solutions besides the trivial solution. Since the system has a nontrivial solution, the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is linearly dependent. For example, let $t = 1$. Then, $c_1 = -10, c_2 = -1, c_3 = 1$. Remember that the matrix equation

$$\begin{bmatrix} 1 & 3 & 13 \\ -1 & 4 & -6 \\ 0 & 7 & 7 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is equivalent to the vector equation $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3} = \boldsymbol{0}$. Plugging in the values for c_1, c_2, c_3, we get $-10\boldsymbol{v_1} - \boldsymbol{v_2} + \boldsymbol{v_3} = \boldsymbol{0}$. Therefore, we have a linear dependence relation between the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$.

LINEAR INDEPENDENCE: EXAMPLE 2

Let's look at a second example.

Determine if the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is linearly independent.

$$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \boldsymbol{v_3} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$$

Form the augmented matrix with $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ as columns:

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 2 & -4 & 0 \\ -1 & 0 & 4 & 0 \end{bmatrix}$$

Let's try to solve this augmented matrix using row operations. Looking at the first and third rows, we can cancel out the leading terms, so let's do $R1 + R3 \longrightarrow R3$:

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 3 & 8 & 0 \end{bmatrix}$$

Looking at the second row, notice the leading coefficient is 2. Let's make that a 1 by doing $\frac{1}{2}R2 \longrightarrow R2$:

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & 8 & 0 \end{bmatrix}$$

Let's get rid of the leading coefficient in the third row by doing $-3R2 + R3 \longrightarrow R3$:

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 14 & 0 \end{bmatrix}$$

From the third row, we know $14c_3 = 0$. So $c_3 = 0$. From the second row, we know $c_2 - 2c_3 = 0$. But $c_3 = 0$. So $c_2 - 2(0) = 0$. Therefore, $c_2 = 0$. From the first row, we know $c_1 + 3c_2 + 4c_3 = 0$. But we know $c_2 = c_3 = 0$. So $c_1 = 0$. Since $c_1 = c_2 = c_3 = 0$, the set of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ is linearly independent.

PROBLEM SET: LINEAR INDEPENDENCE

1. Determine if the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is linearly independent.
   a. $\boldsymbol{v_1} = \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix}$ $\boldsymbol{v_2} = \begin{bmatrix} -1 \\ 0 \\ 8 \end{bmatrix}$ $\boldsymbol{v_3} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$
   b. $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\boldsymbol{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\boldsymbol{v_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
   c. $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\boldsymbol{v_2} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ $\boldsymbol{v_3} = \begin{bmatrix} 7 \\ 7 \\ 9 \end{bmatrix}$

SOLUTION SET: LINEAR INDEPENDENCE

1. Determine if the set {$\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$} is linearly independent.

   a. $\boldsymbol{v_1}=\begin{bmatrix}6\\6\\7\end{bmatrix}$ $\boldsymbol{v_2}=\begin{bmatrix}-1\\0\\8\end{bmatrix}$ $\boldsymbol{v_3}=\begin{bmatrix}2\\2\\-2\end{bmatrix}$

   Form the matrix with the given vectors as columns and the zero column on the right.

$$\begin{bmatrix}6 & -1 & 2 & 0\\6 & 0 & 2 & 0\\7 & 8 & -2 & 0\end{bmatrix} \quad R_1\text{-}R_2\rightarrow R_2$$

$$\begin{bmatrix}6 & -1 & 2 & 0\\0 & -1 & 0 & 0\\7 & 8 & -2 & 0\end{bmatrix} \quad 7R_1\text{-}6R_3\rightarrow R_3$$

$$\begin{bmatrix}6 & -1 & 2 & 0\\0 & -1 & 0 & 0\\0 & -55 & 26 & 0\end{bmatrix} \quad -55R_2+R_3\rightarrow R_3$$

$$\begin{bmatrix}6 & -1 & 2 & 0\\0 & -1 & 0 & 0\\0 & 0 & 26 & 0\end{bmatrix}$$

The third row tells us 26z=0. So z=0. The second row tells us –y=0. So y=0. The first row tells us

6x-y+2z=0. So x=0. The system of equations has only the trivial solution. Therefore, the given vectors are linearly independent.

   b. $\boldsymbol{v_1}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\boldsymbol{v_2}=\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\boldsymbol{v_3}=\begin{bmatrix}1\\1\\0\end{bmatrix}$

   Form the matrix with the given vectors as columns and the zero column on the right.

$$\begin{bmatrix}1 & 0 & 1 & 0\\1 & 0 & 1 & 0\\1 & 1 & 0 & 0\end{bmatrix} \quad R_1\text{-}R_2\rightarrow R_2$$

$$\begin{bmatrix}1 & 0 & 1 & 0\\0 & 0 & 0 & 0\\1 & 1 & 0 & 0\end{bmatrix} \quad R_1\text{-}R_3\rightarrow R_3$$

$$\begin{bmatrix}1 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & -1 & 1 & 0\end{bmatrix} \quad R_2\leftrightarrow R_3$$

$$\begin{bmatrix}1 & 0 & 1 & 0\\0 & -1 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

The second row tells us –y+z=0. Let z=t. Then y=t. The first row tells us x+z=0. So x=-t.

All solutions will of the form $\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, where t is any real number. Since the system has non-trivial solutions, the given vectors are linearly dependent.

c. $\boldsymbol{v_1}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\boldsymbol{v_2}=\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ $\boldsymbol{v_3}=\begin{bmatrix} 7 \\ 7 \\ 9 \end{bmatrix}$

Form the matrix with the given vectors as columns and the zero column on the right.

$$\begin{bmatrix} 1 & 2 & 7 & 0 \\ -1 & -2 & 7 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix} \quad R_1+R_2\rightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 7 & 0 \\ 0 & 0 & 14 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$$

The second and third rows tell us that z=0. The first row tells us x+2y+7z=0. So x+2y=0. Letting y=t, we find x=-2t. So all solutions will be of the form $\begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. There are non-trivial solutions; so the given set of vectors are linearly dependent.

SUMMARY: VECTORS

- We add two vectors by adding their corresponding entries. We multiply a vector by a scalar by multiplying each entry in the vector by the scalar. We multiply matrix to a vector by taking the first row of the matrix and multiplying each entry in the first row by the corresponding entry in the vector, taking the second row of the matrix and multiplying each entry in the second row by the corresponding entry in the vector, etc.

- If we have a set of vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}$ and scalars $c_1, c_2, \dots, c_k$, then $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$ is called a ***linear combination*** of $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}$ and $c_1, \dots, c_k$ are called ***weights***.

- Let $\boldsymbol{v_1}, \dots, \boldsymbol{v_k} \in \mathbb{R}^n$. Then the set $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_k}\}$ is ***linearly independent*** just in case the vector equation $c_1\boldsymbol{v_1} + \cdots + c_k\boldsymbol{v_k} = 0$ has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$. Otherwise, the set is said to be ***linearly dependent***..

4 – MATRIX OPERATIONS

ADDITION AND SCALAR MULTIPLICATION

In this section, we're going to learn about matrix operations. But first, we want to know what a matrix is. An $m \times n$ matrix is an array $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ with m rows and n columns.

We can add two matrices if they have the same size. For example, suppose we want to add the following two matrices:

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 8 \\ 8 & 2 & 3 \end{bmatrix}$$

To add these two matrices, we just add their corresponding entries like this:

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 8 \\ 8 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 8 \\ 9 & 3 & 2 \end{bmatrix}$$

We can also multiply a matrix by a scalar. For example, let's say the scalar is $c = 3$ and the matrix is $A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$. To find cA, we multiply each entry of A by the scalar c:

$$cA = 3\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 12 \\ 0 & 3 & 3 \end{bmatrix}$$

Let's do another example. Suppose $c = -2$ and $A = \begin{bmatrix} 11 & 10 \\ 8 & 9 \\ 4 & 2 \end{bmatrix}$.

$$cA = -2\begin{bmatrix} 11 & 10 \\ 8 & 9 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -22 & -20 \\ -16 & -18 \\ -8 & -4 \end{bmatrix}$$

We can define subtraction of two matrices $A - B$ as $A + (-1)B$. For example, suppose $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. Then we can find $A - B$ as follows:

$$\begin{aligned} A - B = A + (-1)B &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -1 \\ -2 & -4 \end{bmatrix} \end{aligned}$$

MULTIPLICATION

We can also multiply two matrices A and B as long as the number of columns of A is equal to the number of rows of B. Let's do an example. Suppose A is a 2×3 matrix and B is a 3×2 matrix as follows:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$$

The product AB will be a 2×2 matrix.

$$AB = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Notice that the number of columns of A is 3 and it matches up with the number of rows of B. We need this because we're going to multiply each entry in the rows of A with the corresponding entry in the columns of B. For instance, consider the first row of A and the first column of B.

$$\begin{bmatrix} 1 & 3 & -1 \\ & & \end{bmatrix} \begin{bmatrix} 1 & \\ 1 & \\ -1 & \end{bmatrix}$$

The first entry in the first row of A is 1 and it corresponds to the first entry in the first column of B, which is 1. The second entry in the first row of A is 3 and it corresponds to the second entry in the first column of B, which is 1. The third entry in the first row of A is -1 and it corresponds to the third entry in the first column of B, which is -1.

To find the entry of AB in the first row and first column of AB, multiply out the corresponding entries and add: $1 \cdot 1 + 3 \cdot 1 + (-1) \cdot (-1) = 5$. So the product AB looks like this so far:

$$\begin{bmatrix} 5 & \\ & \end{bmatrix}$$

To find the entry of AB in the first row and second column, use the first row of A and the second column of B, and perform the same procedure: $1 \cdot 0 + 3 \cdot 0 + (-1) \cdot 1 = -1$. So we get:

$$\begin{bmatrix} 5 & -1 \\ & \end{bmatrix}$$

Now, move on to the second row of A and the first column of B to get:

$$\begin{bmatrix} 5 & -1 \\ 0 & \end{bmatrix}$$

Finally, move on to the second row of A and the second column of B:

$$\begin{bmatrix} 5 & -1 \\ 0 & 2 \end{bmatrix}$$

What if we had a matrix $C = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix}$, which is a 2×3 matrix. Then we can't multiply AC:

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

The number of columns of A is 3 and the number of rows of C is 2, and they don't match up. So we can't multiply them.

Now, for a general matrix, if the number of rows is the same as the number of columns, then the matrix is called a ***square matrix***. We can multiply two square matrices, of the same dimension, in any order. For example, suppose we had the following 2×2 square matrices:

$A = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$

We can find AB:

$$\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -1 & -1 \end{bmatrix}$$

We can also multiply them in reverse order:

$$BA = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$$

Note that, in this case, $AB \neq BA$. In general, matrix multiplication is not commutative.

PROBLEM SET: MATRIX OPERATIONS

Calculate A+B and A-B.

1. $A = \begin{bmatrix} 3 & 4 \\ 5 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 8 & 0 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 2 \\ 10 & -1 \end{bmatrix}$

Find cA.

1. c=2 A=$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$

2. c=8 A=$\begin{bmatrix} 4 & 3 & 1 \\ 0 & 0 & 2 \\ -3 & -3 & 9 \end{bmatrix}$

Find AB if A and B can be multiplied. Otherwise, indicate why they cannot be multiplied.

1. $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 1 \\ 8 & 7 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -2 & 3 \\ 6 & 6 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 & 1 \\ 7 & 6 & 5 \end{bmatrix}$

SOLUTION SET: MATRIX OPERATIONS

Calculate A+B and A-B.

3. $A+B=\begin{bmatrix}2 & 4\\6 & 2\end{bmatrix}$ $A-B=\begin{bmatrix}4 & 4\\4 & 0\end{bmatrix}$

4. $A+B=\begin{bmatrix}10 & 2\\9 & -1\end{bmatrix}$ $A-B=\begin{bmatrix}6 & -2\\-11 & 1\end{bmatrix}$

Find cA.

3. $cA=\begin{bmatrix}-2 & 0\\-2 & 2\end{bmatrix}$

4. $cA=\begin{bmatrix}32 & 24 & 8\\0 & 0 & 16\\-24 & -24 & 72\end{bmatrix}$

Find AB if A and B can be multiplied. Otherwise, indicate why they cannot be multiplied.

6. $AB=\begin{bmatrix}6 & 6\\1 & -2\end{bmatrix}$

7. $AB=\begin{bmatrix}7 & -2 & 13\\5 & 12 & 3\\64 & 50 & 62\end{bmatrix}$

8. A is 2 by 2 and B is 3 by 3.

9. $AB=\begin{bmatrix}6 & -1 & 5\\2 & -1 & 0\end{bmatrix}$

10. $AB=\begin{bmatrix}7 & 6 & 6\\-7 & -6 & -3\\0 & 0 & 1\end{bmatrix}$

SUMMARY: MATRIX OPERATIONS

- When adding two matrices, we simply add the corresponding entries. When multiplying a matrix by a scalar, we multiply each entry of the matrix by the scalar.

- To multiply two matrices A and B, the number of columns of A has to be the same as the number of rows of B. To find the ij-th entry of AB, we take the entries of the i-th row of A and multiply by the corresponding entries of the j-th column of B, then add the results.

- A square matrix is a matrix with the same number of rows and columns.

CONCLUSION

Congratulations on completing the Linear Algebra book! Here is a review of what we have covered in this course:

- **Gaussian Elimination**
- **Vectors**
- **Matrix Algebra**
- **Determinants**
- **Vector Spaces**
- **Subspaces**
- **Span and Linear Independence**
- **Basis and Dimension**

I hope this book has been useful to you, and I wish you the best in your career and future endeavors. If you feel that you've benefitted from this course, I'd really appreciate it if you wrote a short review for the book.

Be sure to get the companion online course Linear Algebra for Beginners here: https://www.onlinemathtraining.com/linear-algebra/. For more online courses, visit: http://www.onlinemathtraining.com/.

Thank you, again!

Richard Han

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